**Bridging Behavioral and Rational Finance: New Classes of Probability Weighting Functions**  
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**Chapter 4. Probability-Weighting Functions for Variance–Mean Mixtures and Tempered-Stable Families**

**4.3. Probability-Weighting Functions for the Class of Tempered-stable Distributions**

Tempered-stable (TS) laws—also called *exponentially tempered α-stable*, *tempered Lévy-stable*, or, in an important special case, *CGMY/KoBoL*—arise by exponentially damping the Lévy tails of a strictly α-stable process. The exponential tempering preserves the stable-like semi-heavy tails and flexible skew while ensuring all positive moments exist and Fourier-based transforms remain numerically stable. This makes TS models well suited for asset-return dynamics, option pricing, and risk measurement: they interpolate between Gaussian (thin tails) and α-stable (very heavy tails), yet are closed under convolution and provide analytic characteristic functions that calibrate reliably in practice (Koponen, 1995; Carr, Geman, Madan, and Yor, 2002; Boyarchenko and Levendorskii, 2002; Rosiński, 2007; Cont and Tankov, 2004, see also Rachev, Kim, Bianchi, and Fabozzi (2011) for a comprehensive treatment of Lévy models, calibration, and empirical evidence, including tempered-stable specifications).

We use the CGMY/KoBoL parameterization with activity , tempering rates , scale , and location . The Lévy density is

so the exponential factors temper the α-stable power law . The characteristic exponent (under a finite-variation truncation) is

and the characteristic function is . The moment-generating function exists for and equals

***Remark: Lévy–Khintchine representation, Lévy triplet, and relevance for tempered-stable laws***

*Any infinitely divisible distribution on is characterized by its Lévy–Khintchine exponent*

*where is the Lévy triplet: is the drift, the Brownian variance, and the Lévy measure on satisfying  
.  
The indicator in the integrand fixes a truncation convention; choosing a different truncation replaces by for a suitable , leaving the law unchanged. Path properties are read from : finite variation iff and ; activity (finite vs. infinite) from ; tails from the behavior of at .*

*Comprehensive expositions appear in Sato (1999, Chapters 8–9), Cont & Tankov (2004, Chapter 3), and Rosiński (2007) for tempered-stable construction; applications to finance include Carr et al. (2002) and Rachev, Kim, Bianchi, and Fabozzi (2011).*

***Remark: Lévy–Itô decomposition, characteristics, activity, quadratic variation, and moment characteristics from the Lévy measure***

*A càdlàg[[1]](#footnote-1) Lévy process is a stochastically continuous process with stationary, independent increments and . By the Lévy–Itô decomposition (Itô, 1956; Sato, 1999, §19; Applebaum, 2009, §2.4), every Lévy process admits the pathwise representation*

*where is a standard Brownian motion, is a Poisson random measure on with intensity , and is its compensated version for . The triplet is the Lévy triplet: (drift), (Gaussian variance rate), and the Lévy measure satisfying . This decomposition exhibits the continuous martingale part , the compensated small jumps (a square-integrable martingale), and the compound Poisson big jumps. Different truncation functions on yield equivalent decompositions with an adjusted drift .*

*The predictable characteristics of in the semimartingale[[2]](#footnote-2) sense (Jacod & Shiryaev, 2003, §II.2) are with the finite-variation characteristic, the quadratic characteristic of the continuous local martingale part, and the jump compensator. These characteristics determine uniquely and encode its path properties. The activity time (or Blumenthal–Getoor activity index) of the jumps is the smallest such that (Blumenthal & Getoor, 1961). Infinite activity obtains when ; finite activity[[3]](#footnote-3) corresponds to a compound Poisson jump part. Finite variation of the pure-jump component occurs iff and (Sato, 1999, Theorem 21.9).*

*The square-brackets process (quadratic variation) satisfies*

*where is the continuous martingale part and the jumps. Its compensator (predictable quadratic variation) is whenever ; otherwise has infinite expectation driven by the small-jump activity. The decomposition makes clear how volatility accumulates: continuously at rate and discretely through the jump energy .*

*Moment existence and explicit moment characteristics are governed entirely by the Lévy measure near zero (small jumps) and at infinity (tails). For , a necessary and sufficient condition for is*

*with analogous conditions when (Sato, 1999, §25). When the moment generating function exists in a neighborhood of zero, raw moments follow by differentiating the cumulant function ; for Lévy processes, . In particular, for finite first and second moments,*

*under the corresponding integrability conditions. Higher central moments and cumulants are similarly linear in and are explicit integrals of with respect to , thereby making “moments as characteristics” a precise statement: the growth rates of all finite cumulants are given by times the Lévy–Khintchine cumulants determined by (and for order two).*

*For tempered-stable specifications used in §4.3, and . The Lévy–Itô form shows that small-jump activity is infinite when and of finite variation iff , while the exponential tempering ensures all positive moments exist by making for every . Consequently, quadratic variation and higher-order moment growth are explicit and stable to compute, which is crucial for our probability-weighting transforms and for calibration in option pricing.*

Closed-form PDFs for are generally unavailable, but the law is analytically specified by ; densities and CDFs are computed accurately by FFT or fractional-parabolic PDE solvers used in option pricing (Carr et al., 2002; Cont & Tankov, 2004). We write and . Both admit stable numerical evaluation across parameter regimes, and quantiles are obtained by monotone root-finding with .

All raw moments exist. Differentiating at yields the first four cumulants:

Hence the skewness and excess kurtosis are

Skewness sign is controlled by vs. (right-skew when ), while tail thickness rises as (stable-like) and falls as (Gaussian-like). Location shifts the law without changing dispersion or shape. These transparent roles mirror Section 4.2 and will anchor our behavioral “calm/greedy/fearful” regimes when we construct PWFs and higher-order transforms for the TS family.

A diagram of a normal distribution

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**Figure 4.3(a) PDFs of Tempered-Stable with N(0,1) for comparison**

Figure 4.3(a) compares six tempered-stable PDFs to isolate each parameter’s role. A pure location shift () translates the entire curve horizontally with unchanged shape, a benchmark for “prior vs. posterior” comparisons that encode pure re-centering of outcomes without altering dispersion or tail salience. Increasing activity/scale thickens both tails and lowers/widens the peak, assigning more probability to moderate and large moves symmetrically; behaviorally this maps to a “fearful/uncertain” stance that tolerates larger deviations on both sides. Asymmetry enters through the tempering rates: produces right-skew with a longer right tail and mass migrating slightly to positive x, consistent with a “greedy” outlook that stretches upside outcomes and attenuates downside magnitude; inverts the picture, yielding left-skew with a fatter left tail and compressed upper side, reflecting a “fear” orientation that overweights adverse outcomes. Lowering the tail index Y from 1.2 to 0.8 steepens the center and raises tail thickness on both sides; probability moves more gradually across mid-quantiles and persists into the extremes, amplifying tail salience. These channels—location (μ), volatility/scale (), skew via , and tail salience via —will be mirrored in the CDF, quantile, and PWF analyses that follow, where greed bends weighting above the diagonal in mid-quantiles with compressed tails, while fear bends below with elevated tail emphasis.

A graph of a normal distribution

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**Figure 4.3(b) CDFs of Tempered-Stable with N(0,1) for comparison**

Figure 4.3(b) shows the integral behavior implied by the six tempered-stable densities in Figure 4.3(a). The location shift raises the CDF earlier on the left and delays it on the right by an almost constant horizontal translation, confirming a pure re-centering without changing steepness. Increasing the activity/scale flattens the middle portion and widens the –range over which climbs from 0.1 to 0.9; probability accumulates more gradually across the mid-quantiles and reaches the extremes later, reflecting higher dispersion with unchanged symmetry. Right-skew moves upper quantiles outward while pulling lower quantiles inward; the CDF is shallower to the right of the median and steeper to the left, consistent with a greedy orientation that stretches favorable outcomes and attenuates downside magnitude. Left-skew reverses the picture: lower quantiles move farther left and upper quantiles compress to the center; the CDF becomes steeper on the right and flatter on the left, reflecting a fearful stance that overweights adverse outcomes. Lowering the tail index Y from 1.2 to 0.8 sharpens the S-curve at the center while pushing extreme quantiles further out; mass builds more slowly in the tails, increasing the salience of rare events on both sides. In the probability-weighting function analysis, greedy dispositions will bend w(u) above the diagonal in the mid-quantiles and compress tail salience, whereas fearful dispositions bend below with heavier tail emphasis, exactly as these CDF shifts anticipate.

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**Figure 4.3(c) Quantile functions of Tempered-Stable with N(0,1) for comparison**

Figure 4.3(c) depicts the quantile functions for the six tempered-stable specifications together with the standard normal benchmark. The location shift raises the entire curve by an almost constant amount across , confirming a pure translation of outcomes without changing the spacing of quantiles. Increasing the activity/scale compresses the middle of toward the median while sending both extreme quantiles outward; equal probability steps translate into smaller x-increments near and larger increments in the tails, which is the quantile expression of higher dispersion. Asymmetry is read directly from the slope asymmetry of around the center: for right-skew () upper quantiles move farther to the right while lower quantiles approach the center, encoding a greedy orientation that stretches upside opportunities and attenuates downside magnitude; for left-skew () the pattern reverses, with lower quantiles displaced leftward and upper quantiles compressed, reflecting a fearful stance that overweights adverse outcomes. Lowering the tail index thickens both tails: bends more sharply near relative to the normal, while remaining comparatively steeper around the median, signaling greater tail salience together with a tighter central peak. Behaviorally, greed corresponds to quantile spreads that expand on the right and contract on the left, whereas fear expands the left tail and compresses the right; these quantile deformations forecast the probability-weighting patterns we will document in the next figures.

**Probability-weighting functions within the Tempered-Stable family — program of eight cases**

To parallel the Meixner program in §4.2 while capturing distinctive TS mechanics (tempering and activity), we study probability-weighting functions (PWFs) by comparing a benchmark prior CDF in the CGMY/KoBoL class with posteriors that move a single channel or a paired channel. For every case we fix three parameters, vary the fourth (or a targeted pair), and define two posteriors that represent, respectively, a fearful and a greedy disposition. As before, the PWF is

which is invariant to strictly increasing transformations and therefore a canonical device to compare dispositions across models.

Case 1 (scale/volatility channel, ).  
We isolate dispersion changes without altering asymmetry or tail index by varying the activity/scale parameter while keeping fixed. Economically, this is a volatility channel generated purely by jump intensity (with identical tempering on both sides). The fearful posterior uses a higher (more dispersion, thicker central band and fatter tails), while the greedy posterior uses a lower (compressed tails, sharper peak). This case is the TS analogue of §4.2’s scale variation; because all TS moments exist, the PWF signatures are sharp: larger bends below the diagonal in the mid-quantiles and raises tail salience without introducing skew; smaller bends above with compressed extremes.

Case 2 (skew/asymmetry channel, ).  
We vary the left/right tempering asymmetrically by moving the ratio at fixed . Increasing lengthens and thins the right tail relative to the left (positive skew), whereas decreasing does the opposite (negative skew). Behaviorally, right-skew (greedy) stretches upside quantiles and attenuates losses; left-skew (fearful) expands downside quantiles and compresses gains. Because dispersion is nearly preserved around the median, PWFs for this case are largely monotone with no crossing and show pronounced curvature asymmetry above/below .

Case 3 (tail-thickness channel, ).  
We change the Blumenthal–Getoor activity/tail index while holding fixed. Lower increases small-jump activity and thickens both tails; higher thins tails toward Gaussian-like behavior. Fear corresponds to , greed to . The PWF geometry differs from Case 1: mass redistributes into the extremes with little change in the local scale near the center; curves bend most near the shoulders and approach the diagonal around the median.

Case 4 (location channel with shape preserved, ).  
We translate the distribution by varying at fixed . This is a pure re-centering without shape change. The greedy posterior shifts right (), the fearful shifts left (). Although PWFs are defined as probability-of-probability transforms, under a location change they exhibit a single crossing near , producing an almost affine tilt of relative to the 45° line. This case is useful for separating sentiment tilts from risk re-scaling.

Case 5 (joint dispersion–tail channel on a fixed-variance manifold).  
To disentangle variance from tail salience, we move jointly along a curve that keeps constant (using the TS variance formula). Lower is offset by a lower to hold variance fixed, creating “equal-variance” fearful/greedy posteriors that differ only in tail allocation. PWFs now reveal pure tail emphasis: curves depart from the diagonal mostly in the upper and lower deciles with minimal mid-quantile distortion.

Case 6 (quantile-based skew channel with constant variance).  
We mirror §4.2’s quantile-skew design by moving so that the median, variance, and are fixed while the upper and lower decile distances are interchanged. This produces right- vs left-skew posteriors that place equal probability mass at symmetric tail levels. The associated PWFs are nearly linear over the middle quantiles, with curvature concentrated toward the extremes; they furnish a clean diagnostic for skew without a global volatility change.

Case 7 (volatility-adjusted mean channel).  
We vary while simultaneously adjusting so that a volatility-adjusted mean—e.g., a signal-to-noise ratio —is kept at benchmark, fearful, and greedy levels (e.g., ). This separates pure “drift beliefs per unit risk” from raw location shifts. The greedy specification raises and compresses left-tail salience; the fearful lowers , bending below the diagonal while preserving mid-quantile scale.

Case 8 (stress-test channel combining dispersion and skew).  
We move and the skew ratio jointly to mimic flight-to-quality vs search-for-yield regimes: fearful increases and tilts to left-skew; greedy decreases and tilts to right-skew. Tail index and remain fixed. This case captures practical calibration movements observed in option data; PWFs exhibit simultaneous mid-quantile bending (from ) and asymmetric curvature (from ), yielding easily interpretable BF fingerprints.

**Case 1 (TS scale/volatility channel)**

Let the prior (benchmark) be with mean and variance determined by the cumulant formulas in §4.3. The fearful posterior is with ; the greedy posterior is with . All tempering and asymmetry parameters are held fixed, so skew and tail index are unchanged.

Chosen specifications (Case 1: TS scale/volatility channel).  
Prior (benchmark): .  
Fearful posterior: — higher activity/scale.  
Greedy posterior: — lower activity/scale.  
All other parameters are held fixed to preserve symmetry and tail index.

A diagram of a normal distribution

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**Figure 4.3(d) Case 1 – PDFs with standard normal reference**

Figure 4.3(d) displays the three PDFs together with the standard normal as a dashed reference. Increasing shifts probability away from the center and into moderate and large moves on both sides; the fearful density is lower and wider around the mode and decays more slowly than the benchmark but preserves the same semi-heavy tail shape. Decreasing concentrates mass near the center and compresses the tails; the greedy density is taller and narrower than the prior. Relative to the normal, all three TS curves retain thinner centers and heavier shoulders, demonstrating the semi-heavy tail feature produced by tempering stable jumps.

Behavioral–finance reading. In this pure volatility channel, fear corresponds to accepting higher jump activity without asymmetry, which increases the likelihood of crossing both loss and gain thresholds; decision weight effectively moves outward, raising tail salience while diluting mid-quantile mass. Greed corresponds to the opposite stance—reining in jump activity and compressing tail probability, thereby assigning more weight to outcomes near the median and attenuating the impact of extremes.

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**Figure 4.3(e): PWFs for Case 1 (scale/volatility channel)**

Figure 4.3(e) reports the probability–probability transforms for Case 1. The prior law is ; the fearful posterior increases activity to , and the greedy posterior reduces it to with all other parameters fixed. For each we evaluate . The fearful curve lies below the 45° line over the central range and above it near the extremes. Relative to the prior, more mass is allocated to the tails and less to the mid-quantiles, so the posterior probability of being below the prior -quantile is larger when and smaller when ; the graph is concave on and convex on with no crossing. This is the canonical fingerprint of a volatility increase without skew: decision weight shifts outward while tail shape remains unchanged. The greedy specification produces the mirror image. Its curve lies above the diagonal in the interior and approaches the diagonal in the tails, indicating compression of tail salience and a concentration of probability near the center. From a behavioral-finance perspective, fear corresponds to accepting greater dispersion—assigning higher decision weight to tail outcomes on both sides—whereas greed reduces dispersion, effectively underweighting extremes and overweighting moderate outcomes. These PWF signatures isolate the scale channel and will serve as the benchmark for the skew and tail cases that follow.

**Case 2 (skew/asymmetry channel, varying )**

Specification. We fix the activity/scale, tail index, and location at , , and , and move the tempering asymmetry through . The symmetric prior is . The greedy posterior tilts to right-skew with . The fearful posterior tilts to left-skew with . Because and are unchanged, dispersion and tail thickness near the center are nearly preserved; only the allocation across the two tails changes.

A diagram of a normal distribution

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**Figure 4.3(f): PDFs with standard normal reference.**

The prior is symmetric about zero. The greedy specification lengthens and thins the right tail while shortening the left tail; the mode shifts slightly to the right and upper quantiles move outward relative to the prior. The fearful specification mirrors this behavior: the left tail becomes longer and the right tail shorter with a modest mode shift to the left. All three tempered-stable curves retain semi-heavy shoulders compared with the standard normal, but the skewed posteriors concentrate probability asymmetrically across the two sides while maintaining comparable central dispersion.

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**Figure 4.3(g): PWFs for Case 2 (skew/asymmetry channel)**

We compute the probability–probability maps . The greedy right-skew transform lies strictly above the 45° line on and is strongly concave. For lower quantiles the posterior probability of being below the prior -quantile is smaller than , indicating that losses are attenuated; for upper quantiles it is larger than , indicating that gains are amplified. The fearful left-skew transform lies strictly below the diagonal and is strongly convex, reversing the allocation: probabilities of being below prior quantiles are larger for low and smaller for high , consistent with overweighting of adverse outcomes and compression of gains. Unlike the pure scale channel of Case 1, there is no interior crossing and curvature asymmetry is pronounced about ; the mid-quantiles remain close to the diagonal while the deviations concentrate toward the upper and lower deciles. Economically, right-skew provides a “greedy” stance that stretches upside opportunities and reduces the effective weight on downside realizations, whereas left-skew provides a “fear” stance that expands downside salience and trims upside weight without a global change in volatility.

**Case 3 (tail-thickness channel, ).**

Specification. We vary the Blumenthal–Getoor activity/tail index while holding fixed. The symmetric prior is . The fearful posterior thickens tails with , increasing small-jump activity; the greedy posterior thins tails with , moving the law toward Gaussian behavior. Because location and asymmetry are unchanged, dispersion near the center changes only mildly; the main redistribution is between middle mass and the extremes.

A diagram of a normal distribution

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**Figure 4.3(h): Case 3 - PDFs with standard normal reference.**

Lowering raises peak sharpness while pushing probability into both tails; the fearful density decays more slowly than the prior on both sides. Raising compresses tail mass and broadens the center; the greedy density sits closer to the normal at moderate while remaining semi-heavy relative to . The three curves maintain symmetry but differ markedly in shoulder thickness, isolating tail salience without introducing skew.

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**Figure 4.3(i): PWFs for Case 3 (tail-thickness channel).**

The probability–probability transforms display the tail channel’s distinctive geometry. The fearful transform lies below the diagonal over the upper half and above it over the lower half, with curvature concentrated toward the deciles; relative to the prior, the posterior assigns higher probability to being below low prior quantiles and to exceeding high prior quantiles, exactly the hallmark of heavier tails. The greedy transform shows the opposite: it lies above the diagonal in the lower half and below in the upper half, reflecting thinner tails and reduced extreme-event weight while leaving the mid-quantiles comparatively closer to the diagonal than in the scale channel. No interior crossing occurs beyond the inherent symmetry around ; the deviations concentrate near the shoulders, distinguishing Case 3 from Case 1 in which the largest departures occur in the middle ranks.

**Case 4 (location channel with shape preserved, ).**

Specification. We shift the distribution horizontally by varying at fixed . The prior is . The greedy posterior re-centers to the right with , while the fearful posterior re-centers to the left with . Shape—scale, skew, and tail thickness—remains unchanged.

A diagram of a normal distribution

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**Figure 4.3(j):Case 4 - PDFs with standard normal reference.**

All three tempered-stable densities have the same height and thickness of shoulders; the only difference is a rigid translation. Relative to , each curve displays semi-heavy tails due to tempering, but the spacing of quantiles and the modal elevation are invariant across . This isolates sentiment tilts unconfounded by volatility or tail changes.

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**Figure 4.3(k): PWFs for Case 4 (location channel)**

For , a location shift produces an almost affine tilt about the diagonal with a single crossing near . The greedy transform lies above the 45° line for and below it for , reflecting lower posterior probability of falling below low prior quantiles (loss thresholds move right) and higher posterior probability of exceeding high prior quantiles (gain thresholds move left). The fearful transform reverses the tilt, lying below the diagonal for and above for . Curvature is modest compared to Cases 1–3 because only the reference point is moved; there is no mid-quantile bending due to scale or tail reallocation, and no asymmetry-driven curvature. This case therefore cleanly separates “optimism/pessimism” location effects from true risk re-scaling.

**Case 5 (joint dispersion–tail channel on a fixed-variance manifold).**

Specification.

* Prior: TS.
* Fearful (fatter tails + higher dispersion): TS.
* Greedy (thinner tails + lower dispersion): TS.

**A graph of a function

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**Figure 4.3(l): Case 5 PDFs without fixed variance**

The panel in Figure 4.3(l) isolates how joint movements in redistribute mass between the center and the extremes when variance is allowed to adjust. Lowering to while raising to increases small-jump activity and thickens both tails. The density becomes taller at the very mode but also markedly sharper—probability exits the mid-quantiles faster and reappears in the shoulders, producing visibly heavier tails than the prior and the standard normal. Raising to with a much smaller produces the complementary effect: the distribution is narrower, tails decay more rapidly, and mass concentrates around the center. Because the specifications remain symmetric, the changes are purely tail–vs–center reallocations without skew.

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**Figure 4.3(m): PWFs for Case 5 (variance not fixed)**

In the PWFs, the fearful transform bends below the diagonal on the upper half and above on the lower half. For a given prior -quantile , the fearful posterior assigns (i) more probability to being below low quantiles (loss thresholds become easier to cross) and (ii) more probability to exceeding high quantiles (extreme gains become more likely as well). The curve’s S-shape is therefore a clean signature of tail amplification without asymmetry. Conversely, the greedy transform lies above the diagonal in the interior and returns to the diagonal in the far tails: it compresses mid-quantile mass toward the center and reduces the probability of extreme events on both sides. Relative to the pure scale change in Case 1, the curvature here is concentrated nearer the shoulders; relative to the tail-only change with variance held fixed (the earlier Case 5 baseline), relaxing the variance constraint amplifies departures from the diagonal and makes the BF signal visible over the entire range.

**Case 6 (quantile-based skew channel with constant variance).**

Specification. We construct right- vs left-skew posteriors by moving the tempering pair while keeping the tail index , the median/location , and the variance fixed. The prior is . We set for a right-skew posterior and for a left-skew posterior; in each case is rescaled so that  
matches the prior variance. This preserves dispersion and tail index while interchanging upper vs lower tail weight.

A diagram of a normal distribution

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**Figure 4.3(n): Case 6 - PDFs with standard normal reference.**

The prior is symmetric. The right-skew density has a longer, thinner right tail and a shorter, thicker left tail; the left-skew density mirrors this. Because and the variance are fixed, all three densities share comparable central dispersion; differences appear primarily in the shoulders. Relative to the normal, the skewed laws retain semi-heavy tails but allocate mass asymmetrically across the two sides.

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**Figure 4.3(o): PWFs for Case 6 (quantile-skew, constant variance)**

With , the greedy right-skew transform lies strictly above the diagonal and is strongly concave, indicating attenuation of downside probabilities for low and amplification of upside probabilities for high . The fearful left-skew transform lies strictly below the diagonal and is strongly convex, reversing the allocation: losses are overweighted and gains underweighted at matched prior quantiles. Because the median and variance are preserved, the curves are nearly linear through mid-quantiles and the curvature concentrates toward the extremes, giving a clean diagnostic of skew without confounding volatility changes.

**Case 7 (volatility-adjusted mean channel).**

Specification. We vary while adjusting so that a volatility-adjusted mean (signal-to-noise) is fixed at benchmark, fearful, and greedy levels. We keep and constant. Using , we set and choose to obtain . The resulting specifications are:

* Benchmark: with .
* Fear: with (higher volatility for the same ).
* Greed: with (lower volatility for the same ).

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**Figure 4.3(p): Case 7 - PDFs with standard normal reference.**

All three densities share the same mean sign but differ in dispersion by construction. The greedy specification concentrates probability near the mean and shortens both tails; the fearful specification spreads mass outward and thickens the shoulders. Relative to , each tempered-stable density still exhibits semi-heavy tails, yet the volatility scaling induced by makes the relative tail prominence markedly different across the three curves. Because are fixed, these are genuine changes in “drift per unit risk,” not in skew or tail index.

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**Figure 4.3(q): PWFs for Case 7 (volatility-adjusted mean channel)**

The P–P transforms reflect the signal-to-noise tilt. With (greed), the curve rises steeply above the diagonal, indicating that for any prior quantile threshold losses are less likely to be breached and high quantiles are more likely to be exceeded; curvature is strongest in the mid-to-upper quantiles where volatility compression matters most. With (fear), the curve lies below the diagonal across most of the range, reflecting increased likelihood of falling below prior thresholds and reduced likelihood of surpassing high ones. Unlike Case 4’s pure translation, curvature here is not an affine tilt: the diagonal crossing occurs near the median but the shape is S-like because the volatility adjustment modulates tail salience while the mean is held fixed in absolute terms.

**Case 8 (stress-test channel combining dispersion and skew).**

Specification. We move the scale parameter and the skew ratio jointly to mimic flight-to-quality vs search-for-yield regimes, holding and fixed. The prior is . The fearful posterior raises activity and tilts to left-skew, ; the greedy posterior lowers activity and tilts to right-skew, .

A graph of a normal distribution

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**Figure 4.3(r): Case 8 - PDFs with standard normal reference.**

Relative to the symmetric prior, the fearful density shifts mass into the left tail while broadening dispersion overall; the mode drops and the right shoulder shortens. The greedy density concentrates mass near the center and stretches the right tail while compressing the left. All three retain semi-heavy tails vis-à-vis , but the combined scale–skew moves generate visibly different asymmetries and volatility levels consistent with stress vs calm market states.

A graph of a stress-strain curve

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**Figure 4.3(s): PWFs for Case 8 (scale-skew stress test)**

The probability–probability transforms display simultaneous mid-quantile bending (from ) and asymmetric curvature (from ). The greedy transform lies above the diagonal across the interior with strong concavity, signaling both volatility compression and upside tilt—loss thresholds are harder to breach and high quantiles are easier to exceed. The fearful transform lies well below the diagonal over most of with a pronounced convex shape and a steep upturn only near ; this reflects elevated dispersion and left-skew, making downside thresholds easy to cross and attenuating the chance to clear high quantiles. In contrast to Cases 1–2 where scale or skew acts in isolation, Case 8 produces BF fingerprints that combine the signatures: center-overweighting plus upside preference under greed; center-underweighting plus downside salience under fear.

**Concluding remarks.**

This section developed a systematic program for constructing and interpreting probability-weighting functions (PWFs) within the tempered-stable (TS/CGMY–KoBoL) class. The analysis proceeded in eight calibrated cases that isolate economically meaningful channels—scale, skew, tail thickness, location, tail/scale trade-offs under variance constraints, quantile-based skew at constant variance, volatility-adjusted mean, and a combined scale–skew stress test. Across these designs, TS laws proved to be an analytically tractable and empirically faithful platform for linking distributional primitives to behavioral-finance dispositions. Several robust findings emerge.

First, the TS family furnishes clean separations between channels because the characteristic exponent encodes them in distinct parameters: the activity/scale constant governs overall dispersion; the tempering pair governs asymmetry; the Blumenthal–Getoor index governs tail salience while preserving semi-heavy tails; and controls location. The cumulant formulas obtained from the Lévy–Khintchine exponent yield closed-form expressions for the first four moments, which in turn clarify how incremental moves in map into changes in variance, skewness, and kurtosis. This transparency underlies the stability of our numerical evaluation of PDFs, CDFs, and quantiles, and it explains the high numerical quality of the PWFs and their higher-order primitives.

Second, the geometry of the PWFs retains crisp and interpretable fingerprints that are invariant to monotone transformations of payoffs. Pure scale moves (Case 1) bend below the 45° line over the interior when dispersion rises and above it when dispersion falls, while approaching the diagonal in the far tails because tail shape is unchanged. Skew moves (Case 2) induce strongly concave or convex transforms with little mid-quantile displacement, revealing that asymmetry reallocates probability between gains and losses without a global change in volatility. Tail-thickness moves (Case 3) push curvature toward the shoulders: heavier tails pull below the diagonal at high and above at low , whereas thinner tails reverse that pattern. Location shifts (Case 4) are distinguished by an almost affine tilt with a single crossing near the median—an effect that is easily separated from genuine changes in risk because curvature is minimal. These canonical shapes persist under a variety of parameterizations and provide diagnostic templates that practitioners can recognize visually.

Third, by moving along economically motivated manifolds we showed how to sharpen inference. The fixed-variance tail channel (Case 5) demonstrated that holding constant forces and to offset, which pushes PWF deviations into the deciles and yields nearly linear mid-quantile behavior. Relaxing the variance constraint reveals the full tail signal and produces visibly stronger curvature, but the diagnostic—deviations concentrated away from the center rather than uniformly—is unchanged. Likewise, the quantile-skew design at constant variance (Case 6) preserves the median and dispersion while interchanging upper and lower decile distances. The resulting PWFs are nearly linear in the middle with curvature concentrating toward the extremes, giving a clean skew diagnostic uncontaminated by volatility changes. These designs are practically valuable for model calibration: they align one-to-one with common identification targets in option smiles (wings for , slope asymmetry for , and at-the-money width for ) and with distributional features in high-frequency return data.

Fourth, the volatility-adjusted mean channel (Case 7) separates beliefs about drift per unit risk from raw location moves. Adjusting so that attains benchmark, fearful, and greedy levels yields PWFs with pronounced S-shapes rather than the affine tilts of Case 4. Greed compresses left-tail salience and lifts the curve above the diagonal for most ; fear broadens dispersion relative to mean and bends the curve below the diagonal, especially over the interior where signal-to-noise matters most. This construction is useful in empirical work: many economic narratives—policy regimes, carry trades, and risk-parity reallocations—are naturally stated in terms of Sharpe-like quantities rather than raw drift.

Fifth, the scale–skew stress test (Case 8) replicates flight-to-quality versus search-for-yield regimes by moving and jointly. Here the PWFs combine the mid-quantile bending of Case 1 with the asymmetric curvature of Case 2, producing fingerprints that are both strong and intuitive: under stress, curves lie well below the diagonal with pronounced convexity and only recover near ; in calm, curves rise decisively above the diagonal with persistent concavity. This joint channel matches the co-movements typically observed in option data and in realized-volatility/skewness measures during macroeconomic shocks, and it provides a compact behavioral interpretation of those episodes.

Methodologically, the section shows that the PWF framework offers a unifying language for mapping parametric return models to behavioral dispositions without privileging a particular payoff scale. Because PWFs compare probability to probability via , the analysis is robust to monotone payoff transformations and thus directly comparable across assets and horizons. Numerically, FFT-based inversion for TS characteristic functions is reliable across our parameter ranges, and quantiles can be computed rapidly with monotone interpolation; these properties allow repeated transforms, grid searches, and stress tests at negligible computational cost.

Economically, the comparative statics of the eight cases produce a coherent taxonomy of “calm/greedy/fearful” regimes. Rising or falling raises tail salience symmetrically and pushes decision weight away from the center; rising or falling produces asymmetric reallocations that favor gains or losses; location and signal-to-noise channels tilt probabilities with distinctive shapes that are easily distinguished from the volatility and tail mechanisms. Because all positive moments exist for TS laws and closed-form cumulants are available, the same specifications that drive PWF behavior also deliver analytically tractable risk measures, option values, and higher-order dominance diagnostics. This alignment strengthens the case for TS models as a bridge between rational-pricing disciplines and behavioral perspectives.

Two limitations merit emphasis. First, while our designs fix a single channel at a time (or a controlled pair), real markets often move along higher-dimensional paths—for example, volatility and tail index co-vary with leverage constraints or liquidity. The stress-test case partially addresses this by combining scale and skew, but richer joint dynamics are relevant for intraday data and for multi-asset settings. Second, the mapping from PWF fingerprints to unique parameter changes is not strictly injective at finite sample sizes: different small perturbations can produce visually similar curves near the diagonal. Identification should therefore incorporate auxiliary statistics (option wings, realized volatility, signed jump measures) in addition to PWF shapes.

These caveats notwithstanding, the section establishes a practical recipe. Start from a symmetric, economically interpretable benchmark calibrated to central moments; move one channel at a time using the closed-form cumulants to maintain or relax invariants (variance, median, signal-to-noise) as needed; compute PWFs and their primitives; and read the resulting shapes against the canonical templates documented here. The procedure yields parameter movements that can be assigned clear behavioral labels, and it scales seamlessly to out-of-sample backtesting, to portfolio overlay decisions, and to option-implied stress testing.

In sum, tempered-stable processes provide a single, parsimonious family in which dispersion, asymmetry, and tail behavior can be manipulated independently yet evaluated consistently through probability-weighting transforms. The eight-case program makes the mapping from model parameters to behavioral interpretations explicit and testable. It shows how risk perceptions—about volatility, asymmetry, and rare events—appear as simple geometric signatures on , and how those signatures persist across numerical implementations and calibration choices. This tight coupling between analytic tractability, empirical relevance, and behavioral interpretability is the principal contribution of the section and a foundation for the applications developed in the remainder of the chapter.

1. Càdlàg (from French *continue à droite, limite à gauche*) describes a real-valued function or process whose paths are:

   Right-continuous at every time : .

   With left limits at every time : the limit exists (though it may differ from ).

   Thus, a càdlàg path can jump, but only via instantaneous upward/downward jumps; no left discontinuities or oscillations at a point. This is the standard path regularity for Lévy and general semimartingale processes. [↑](#footnote-ref-1)
2. A càdlàg adapted process on a filtered probability space is a semimartingale if it can be written as

   where is a local martingale and is a finite-variation adapted process (both càdlàg). Equivalently, semimartingales are precisely the processes that admit a well-defined Itô stochastic integral for all bounded predictable . This is the largest class stable under stopping, localization, and stochastic integration—hence the natural domain for modern stochastic calculus.

   Main examples.

   Brownian motion and Itô diffusions: .

   Lévy processes: any Lévy process jumps is a semimartingale; decomposition is its Lévy–Itô form.

   Includes compound Poisson, Variance Gamma, CGMY/KoBoL, Normal Inverse Gaussian, etc.

   Jump–diffusions (Itô semimartingales): solutions to SDEs driven by Brownian motion and Poisson/random measures:

   Finite-variation processes: any adapted càdlàg finite-variation process (deterministic integrators, cumulative trading costs).

   Special semimartingales: processes with Doob–Meyer decomposition where is predictable (e.g., submartingales with compensators).

   Locally absolutely continuous time changes of the above preserve the semimartingale property.

   Non-examples: fractional Brownian motion with Hurst (in its natural filtration) is not a semimartingale; neither are processes requiring pathwise (Young/rough) integration rather than Itô calculus. [↑](#footnote-ref-2)
3. Activity refers to how many jumps occur in any finite time interval.

   Finite activity: the Lévy measure has finite total mass, .  
   Equivalently, the jump part is compound Poisson: on every interval there are only finitely many jumps a.s. Paths look like a Brownian drift/diffusion plus occasional isolated jumps. Total variation of the jump part is finite iff also .

   Infinite activity: the Lévy measure has infinite mass, .  
   Then there are infinitely many jumps on every a.s., typically with most jumps very small. Paths have a “granular” look—accumulations of tiny jumps possibly superimposed on diffusion. Finite/infinite variation is separate: it depends on (finite ⇒ finite variation of the pure-jump part; infinite ⇒ infinite variation).

   Examples: compound Poisson (finite activity); Variance Gamma and CGMY/KoBoL with (infinite activity; finite variation if , infinite variation if ); Brownian motion has no jumps (jump activity zero). [↑](#footnote-ref-3)